Spring 2011

1

If we call Ed's original amount of pie x and call Anh's original amount y. We are given

$$x - .2x = 2(y - .4y)$$

and we want to solve for x in terms of y.

$$.8x = 2(.6y)$$

$$.8x = 1.2y$$

$$x = 1.5y = 150\%\ of\ y$$

So the answer is **D. 150**

2

We are given that $(a\#b)\#3 = 250$, and working from the definition of $a\#b$ we can translate this into more familiar terms. Starting inside the parentheses,

$$(ab^2 + a)\#3 = 250$$

$$(ab^2 + a) * 3^2 + (ab^2 + a) = 250$$

Now we solve:

$$9(ab^2 + a) + ab^2 + a = 250$$

$$10(ab^2 + a) = 250$$

$$(ab^2 + a) = 25$$

$$a(b^2 + 1) = 25$$

So b has to be a positive integer such that squaring it and adding 1 gets a factor of 25. The only number with that property is 2, which squared and incremented by 1 becomes 5. since $b^2 + 1 = 5$, a must be 5. So $a + b = 2 + 5 = 7$.

So the answer is **B. 7**

3

Let's try to list the ways she can climb 10 steps.

with 10 climbs of 1: 1 way

with 8 1's and a 2: 9 ways (the 2 can be any of the 9 climbs) $\binom{9}{1}$

with 6 1's and 2 2's: 28 ways (the first 2 can be any of the 8, the second any of the 7, giving 56, but half of them are just swapped around, so 28.) $\binom{8}{2}$

with 6 1's and a 4: 7 (the 4 can be any of the 7) $\binom{7}{1}$

with 4 1's and 3 2's: 35 (the 1st 2 has 7 choices, the 2nd has 6, the 3rd has 5, and each order is repeated 3*2 times) $\binom{7}{3}$

with 4 1's, a 2, and a 4: 30 (the 4 has 6 choices then the 4 has 5 left, with no repetitions) $\binom{6}{1}\binom{5}{1}$

with 2 1's and 4 2's: 15 (the first 1 has 6 choices, the second has 5, half are repeats) $\binom{6}{2}$

with 2 1's, 2 2's, and a 4: 30 (the 4 has 5 choices, then the 1st 2 has 4 and the second 2 has 3 with half of those being repeats) $\binom{5}{1}\binom{4}{2}$

with 2 1's and 2 4's: 6 (the 1st 4 has 4 choices, the second has 3, and half are repeats) $\binom{4}{2}$

with 5 2's: 1

with 3 2's and a 4: 4 (the 4 has 4 choices) $\binom{4}{1}$

with a 2 and 2 4's: 3 (the 2 has 3 choices) $\binom{3}{1}$

Total: $1 + 9 + 28 + 7 + 35 + 30 + 15 + 30 + 6 + 1 + 4 + 3 = 169$

So the answer is **E. 169**

4

The sum of 6 consecutive positive integers beginning at n is $n + n + 1 + n + 2 + n + 3 + n + 4 + n + 5 = 6n + 15$, which we want to be a perfect cube, so $6n + 15 = x^3$. This gives us $\frac{x^3-15}{6} = n$. So x^3 has to be 15 more than a multiple of 6, which means it has to be a multiple of 3 but not of 2. After 27 (gotten by $n = 2$), the next 2 perfect cubes that are multiples of 3 but not of 2 are 729 (9^3) and 3375 (15^3). $\frac{729-15}{6} = 119$ and $\frac{3375-15}{6} = 560$, and $119 + 560 = 679$

So the answer is **A. 679**

5

The formula for the sum of an infinite geometric series with first term a and common ratio r is $S = \frac{a}{1-r}$. So we are given $\frac{a}{1-r} = 6$ and since squaring every term squares both the first term and the ratio, $\frac{a^2}{1-r^2} = 15$. We are trying to find the sum of the sequence with first term a and common ratio $-r$, which is $x = \frac{a}{1+r}$. Noticing that that multiplied by S is $\frac{a^2}{1-r^2} = 15$, we know that $6x = 15$, so $x = 2.5$

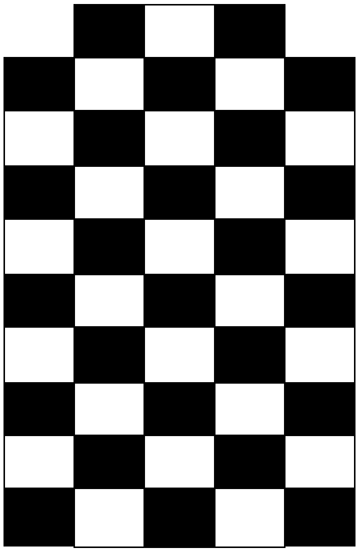
So the answer is **B 2.5**

6

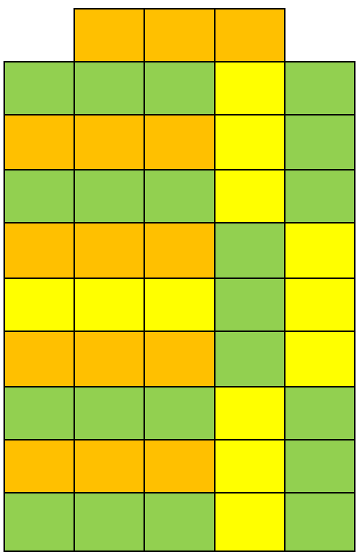
The median of the original set of ten numbers is 6, so the average of the 5th and 6th numbers (assume the numbers are organized by order) is 6. Since adding the number 15 to the set changes the median to the 6th number of the now 11-number set, the 6th number is 8. This means the 5th number is 4. Since 7 is between 4 and 8, it would be the new 6th number and therefore the median if it is inserted rather than 15.

So the answer is **E. 7**

7.



Note that any 2 x 1 rectangle must cover one shaded square and one unshaded square, yet there are 23 unshaded squares and 25 shaded squares, so 24 1 x 2 rectangles cannot cover the shape.



Here is one way that 16 1x3 rectangles can cover the shape.

Since 2 x 1 rectangles cannot cover the shape and both 4 x 1 rectangles and 2 x 3 rectangles can be made up of 2 x 1 rectangles, neither type can cover the shape. So only the 1 x 3 rectangles work.

So the answer is **B. 1**

8

Since side SL is the longest side, for the triangle to be obtuse it must be that

$$|\overline{SM}|^2 + |\overline{ML}|^2 < |\overline{SL}|^2$$

Thus, $SL > \sqrt{12^2 + 17^2} = \sqrt{144 + 289} = \sqrt{433} \approx 20.8$

The smallest integer SL can be is thus 21.

So the answer is **C. 21**

9

P(x) is a polynomial with positive coefficients so it should have the form $a + bx + cx^2 + dx^3 + ex^4 + fx^5$ …, with a,b,c…≥ 0.

Since $P(0) = 3, a + b(0) + c(0) + d(0) + e(0) \ldots = a = 3$

Since $P(3) = 144, a + 3b + 9c + 27d + 81e + 243f \ldots = 144$, so anything beyond and including f is 0. This brings us down to 4 unknown variables and 3 equations. Also, e can only be either 0 or 1.

$$P(1) = 3 + b + c + d + e = 8$$

$$P(2) = 3 + 2b + 4c + 8d + 16e = 39$$

$$P(3) = 3 + 3b + 9c + 27d + 81e = 144$$

We need $P(-2) = 3 - 2b + 4c - 8d + 16e$

If $e = 1$, these equations simplify to

$$I \qquad b + c + d = 4$$

$$II \qquad 2b + 4c + 8d = 20$$

$$III \qquad 3b + 9c + 27d = 60$$

$$\frac{(II)}{2} - (I) \qquad c + 3d = 6$$

$$\frac{III}{6} - \frac{I}{2} \qquad c + 4d = 8$$

$$d = 2$$

$$c = 0$$

$$b = 2$$

This works and gives us $P(x) = 3 + 2x + 2x^3 + x^4$ so $P(-2) = 3 - 4 - 16 + 16 = -1$

So the answer is **E. -1**

10

The 1st three digits should make up the smallest three-digit prime number possible with 1's, 2's, and 3's, which is 113. This makes the next set 13_, and the smallest prime number of that form is 131, giving 31_ for the next number meaning 311. Now we're at 11_ which will give us a cycle. So the number is 1131131131 and the final 2 digits are 31

So the answer is **E. 31**

11

The geometric sequence will have terms a, ar, ar^2, a^3. The arithmetic sequence will have $b, b + d, b + 2d, b + 3d$.

Multiplying them we have $$ab = 96$$

$$abr + adr = 180$$

$$abr^2 + 2adr^2 = 324$$

$$abr^3 + 3adr^3 = 567$$

We can rewrite some of these to get rid of some variables.

$$abr + adr = 180$$

$$r(abr + adr + adr) = 324$$

$$r(180 + adr) = 324$$

$$r(180 + 180 - abr) = 324$$

$$-96r^2 + 360r - 324 = 0$$

$$r = \frac{-360 \pm \sqrt{360^2 - 4(96)(324)}}{2(-96)} = 2.25 \; or \; 1.5$$

Through a similar method:

$$r(r(abr + adr + adr) + adr^2) = 567$$

$$r(324 + adr^2) = 567$$

$$r(324 + 324 - 180r) = 567$$

$$180r^2 - 648r + 567 = 0$$

$$r = \frac{648 \pm \sqrt{648^2 - 4(180)(567)}}{2(180)} = 2.1 \text{ } or \text{ } 1.5$$

Since r is only one number it must be 1.5

$$abr + adr = 180$$

$$96(1.5) + 1.5ad = 180$$

$$ad = 24$$

We need $abr^4 + 4adr^4 = 96(1.5)^2 + 4(24)(1.5)^4 = 972$

So the answer is **B. 972**

12.

$$\log_x y + \log_y x = 2.9$$

$$\frac{1}{\log_y x} + \log_y x = 2.9$$

$$1 + (\log_y x)^2 = 2.9 \log_y x$$

$$(\log_y x)^2 - 2.9 \log_y x + 1 = 0$$

$$\log_y x = \frac{2.9 \pm \sqrt{8.41 - 4}}{2} = \frac{2.9 \pm \sqrt{4.41}}{2} = \frac{2.9 \pm 2.1}{2} = 2.5 \text{ } or \text{ } .4$$

We can choose either so let $\log_y x = 2.5$

$$y^{2.5} = x = \frac{128}{y}$$

$$y^{\frac{7}{2}} = 128$$

$$y = 4, x = \frac{128}{4} = 32$$

$$32 + 4 = 36$$

So the answer is **B. 36**

13.

$$a^5 + b^2 + c^2 = 2011$$

$$c = \sqrt{2011 - a^5 - b^2}$$

Using a TI-83, hit y= then enter $\sqrt{2011 - 1^5 - X^2}$ in y_1 and $\sqrt{2011 - 2^5 - X^2}$ in y_2.

This doesn't get us anything so try $\sqrt{2011 - 3^5 - X^2}$ and $\sqrt{2011 - 4^5 - X^2}$

This gets us 3, 2, 42, which has 2, 3 prime and 42 as the nonprime.

So the answer is **C. 42**

14.

Still using a TI-83, put in y= X/17

Hit trace, then type 1001 enter, 1111 enter, etc. looking for whole numbers from all 90 palindromes

2992, 3553, 4114, 7667, and 8228 are divisible by 17

So the answer is **C. 5**

15.

If we could only use 0 through 5, there would only be 5 possibilities:

0	1	2		0	1	3		0	1	4		0	2	3		0	2	4
3	4	5;		2	4	5;		2	3	5;		1	4	5;		1	3	5

Adding in a 6 allows us to increase 1, 2, 3, 4, 5, or all values by 1 (starting with the highest and working down) so we have 5 * 7 = 35 possibilities

So the answer is **D. 35**

16.

$$a_3 = a_2 + a_1$$

$$a_4 = a_3 + a_2 = 2a_2 + a_1$$

$$a_5 = 3a_2 + 2a_1$$

$$a_6 = 5a_2 + 3a_1$$

$$a_7 = 8a_2 + 5a_1 = 160$$

So a_2 is a multiple of 5 and a_1 is a multiple of 8, and $a_2 > a_1$, so $a_2 = 15, a_1 = 8$

$$a_8 = 13a_2 + 8a_1 = 13(15) + 8(8) = 259$$

So the answer is **C. 259**

17.

We need some number that divides both n+4 and $n^2 + 7$. Since $4^2 + 7 = 23$, if 23 divides $n + 4$ it will also divide $n^2 + 7$, so the first n is 19 (23-4) and they occur every 23 from there. 19+23(86)=1997, which is as high as we can get without going over. So we have 19+23(0) through 19+23(86) for 87 values of n.

So the answer is **C. 87**

18.

Let's start by looking at the biggest pile and working our way down. The probability that we have an odd number of pennies after picking only from the biggest pile is 1/21. Now let's look at what happens when we add in the next pile. We can get an odd number by having an even number already and picking a penny ($\frac{20}{21} * \frac{1}{19} = \frac{20}{19*21}$) or by having an odd number and picking a dime ($\frac{1}{21} * \frac{18}{19} = \frac{18}{19*21}$). This gives a total of $\frac{18+20}{19*21} = \frac{38}{19*21} = \frac{2}{21}$ at this point. Using this strategy, we will continue through each pile:

P(was odd)	P(this pile is a penny)	P(old even, penny)	P(old odd, dime)	P(ends odd)
2/21	1/17	$\frac{19}{21} * \frac{1}{17} = \frac{19}{17*21}$	$\frac{2}{21} * \frac{16}{17} = \frac{32}{17*21}$	$\frac{19+32}{17*21} = \frac{3}{21}$
3/21	1/15	$\frac{18}{21} * \frac{1}{15} = \frac{18}{21*15}$	$\frac{3}{21} * \frac{14}{15} = \frac{42}{21*15}$	$\frac{60}{21*15} = \frac{4}{21}$
4/21	1/13	$\frac{17}{21} * \frac{1}{13} = \frac{17}{13*21}$	$\frac{4}{21} * \frac{12}{13} = \frac{48}{21*13}$	$\frac{65}{21*13} = \frac{5}{21}$

By this point we should see the pattern: when taking n piles, we have an $\frac{n}{21}$ chance of getting an odd number of pennies. We have 10 piles, so we should have a 10/21 chance of an odd number.

So the answer is **A. 10/21**

19.

The sum of each set needs to be as small as possible to get as many sets as possible, and there isn't any way to avoid getting a set with at least 15. We can split the sets as follows: $\{15\} \cup \{1, 14\} \cup \{2, 13\} \cup \{3, 12\} \cup \{4, 11\} \cup \{5, 10\} \cup \{6, 9\} \cup \{7, 8\}$ for 8 sets

So the answer is **D. 8**

20.

We have 4 prime numbers and 5 nonprime numbers. If we let a denote a non-prime and b denote a prime, then we only need to know how many permutations of 4 b's and 5 a's have no b's next to each other. The string bababab is the least possible buffering between b's and leaves 2 a's that can be placed anywhere. We can place each in 5 places (beginning, end, between any two b's). There are 5 ways we can place them in the same place and ${}_5C_2 = 10$ ways we can place them in different spots for a total of 15. There are ${}_9C_4 = 126$ total ways to place 4 b's and 5 a's for a probability of $\frac{15}{126} = \frac{5}{42}$.

So the answer is **C. 5/42**